Lecture 1: Algebraic Varieties

Affine Varieties

$A = k[x_1, \cdots, x_n] \supseteq T$

$Z(T) = \{ (x_1, \cdots, x_n) \in k^n \mid f(x_1, \cdots, x_n) = 0,\ \forall f \in T \}$ algebraic set

- $Z(T_1) \cup Z(T_2) = Z(T_1 T_2)$
- $\cap Z(T_\lambda) = Z(\cup T_\lambda)$.
- $Z(\phi) = k^n$, $Z(1) = \phi$

Zariski topology: closed subsets formed by algebraic sets

ex. Proper closed algebraic subsets of A^1_k are finitely many points from weak Hilbert's Nullstellensatz

(affine) Algebraic Variety is an irreducible algebraic subset of A^n_k

This is the main targets to study in algebraic geometry

Restrict to loci defined by polynomials, one has better tools

to study singularities.

Every Notherian topological space algebraic set can be expressed as union of irreducible subsets algebraic varieties

Dictionary between algebraic sets & algebras

- $T_1 \subseteq T_2$ in A $\Longrightarrow$ $Z(T_1) \supseteq Z(T_2)$
- $Y_1 \subseteq Y_2$ in A^n_k $\Longrightarrow$ $I(Y_1) \supseteq I(Y_2)$

  $I(Y) = \{ f \in A \mid f(y) = 0,\ \forall y \in Y \}$

- (Hilbert's Nullstellensatz) $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, for all $\mathfrak{a} \triangleleft A$

  k: algebraically closed which we will assume from now on

  $\sqrt{\mathfrak{a}} = \{ f \in A \mid f^r \in \mathfrak{a},\ \text{for some } r \in \mathbb{N} \}$

- $Y \subseteq A^n$, $\overline{Y} = Z(I(Y))$

- $\underset{\text{closed}}{Y \subseteq Z(I(Y))} \Rightarrow \overline{Y} \subseteq Z(I(Y))$
- $Y \subseteq \underset{\text{closed}}{Z = Z(\mathfrak{a})} \Rightarrow I(Y) \supseteq I(Z(\mathfrak{a})) \supseteq \mathfrak{a}$
  $Z(I(Y)) \subseteq Z(\mathfrak{a})$

Sketch proof of <u>Hilbert's Nullstellensatz</u>:

- <u>Zariski lemma</u>: K field, finitely generated associative algebra $/k$
  $\Rightarrow K$: finite field extension of k.

- (Weak Hilbert's Nullstellensatz)
  ① $\mathfrak{m} \underset{\text{maximal}}{\triangleleft} A$, then $\mathfrak{m} = (x_1 - a_1, \cdots, x_n - a_n)$, for some $a_i \in k$
  ② $f_i \in A$ w/o common zeros, then $(f_i) = A$
  i.e a proper ideal always has a common zero

pf: ① $A/\mathfrak{m}$ field, finitely generated $/k$ $\xRightarrow{\text{Zariski lemma}}$ $A/\mathfrak{m}$ finite extension of k

$A/\mathfrak{m} \ni x_i$
$\parallel$ $\downarrow$ $\rightarrow \mathfrak{m} \ni x_i - a_i$
$k \ni a_i$

k: algebraically closed

② $(f_i) \subseteq \mathfrak{m}$: maximal ideal but then $(a_1, \cdots, a_n)$ common zero $\rightarrow\!\!\!\!\times$
$\parallel$ ①
$(x_1 - a_1, \cdots, x_n - a_n)$

Proof of Nullstellesatz:
Assume that g vanishes on $Z(f_1, \cdots, f_n)$. goal: $g^r \in (f_1, \cdots, f_n)$
then $f_1, \cdots, f_n$, $x_{n+1} g - 1$ have no comon zero in k^{n+1}

$\xRightarrow{\text{weak Nullstellesatz}}$ $\exists\, p_1, \cdots, p_{n+1}$ polynomials in $k[x_1, \cdots, x_{n+1}]$

$$p_1 f_1 + \cdots + p_n f_n + p_{n+1}(x_{n+1} g - 1) = 1$$

$x_{n+1} = \frac{1}{g}$

$$1 = p_1\left(x_1, \cdots, \tfrac{1}{g}\right) f_1 + \cdots + p_n\left(x_1, \cdots, \tfrac{1}{g}\right) f_n$$

multiply both sides by suitable power of g

Remark: The assumption "k algebraically closed" is crucial.

ex. $k=\mathbb{R}$, then $Z(x^2+y^2+1=0)=\phi$.

Morphism / Functions on Affine Varieties

Definition: ① $Y\subseteq \mathbb{A}_k^n$ variety, $f: \mathbb{A}_k^n \longrightarrow k$ is a regular function

if $\exists\, \underset{\text{open}}{U\subseteq Y}$ $f|_U = g/h$, $g, h \in A$

② $X \xrightarrow{f} Y$ morphism between varieties

if $\forall\, \underset{\text{open}}{V\subseteq Y}$, $V\xrightarrow{g} k$ regular function

then $f^{-1}(V) \xrightarrow{f} V \xrightarrow{g} k$ is a regular function.

Remark: Regular functions are continuous if k equipped w/ Zariski topology

(Grothendieck) Study the ring of functions of a variety to reflect the geometry of a variety

Theorem: $Y\subseteq \mathbb{A}_k^n$ affine variety, then

① $\underset{\text{ring of regular functions}}{\mathcal{O}(Y)} \cong k[x_1,\cdots,x_n]/I(Y) =: A(Y)$

② $p\in Y \overset{1:1}{\longleftrightarrow}$ maximal ideal of $\mathcal{O}(Y)$ $\mathfrak{m}_p$

③ $\mathcal{O}_p \cong A(Y)_{\mathfrak{m}_p}$, $\dim \mathcal{O}_p = \dim Y$

④ $K(Y) \cong$ quotient field of $A(Y)$

function field of Y

$\| $

$\{(U,f) \mid \underset{\text{open}}{U\subseteq Y},\ U \xrightarrow[\text{regular}]{f} k\}$

$\rightsquigarrow$ invariant under birational transformation

Projective Varieties

$S = \mathbb{k}[x_1, \cdots, x_{n+1}] = \bigoplus_{d \geq 0} S_d$ graded ring

$\mathbb{P}^n_{\mathbb{k}} := \left\{ (a_1, \cdots, a_{n+1}) \in \mathbb{A}^{n+1}_{\mathbb{k}} \setminus \{(0, \cdots, 0)\} \right\} / \sim, \quad (a_1, \cdots, a_{n+1}) \sim (\lambda a_1, \cdots, \lambda a_{n+1})$

$T \subseteq S$ subset of homogeneous polynomials

Definition: $Y \subseteq \mathbb{P}^n_{\mathbb{k}}$ is an algebraic set if $Y = Z(T)$, for some T

HW 1.11, 2.12, 2.14